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THE LINEAR PROBLEM OF A VIBRATOR IN A SUBSONIC BOUNDARY LAYER*

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The subsonic flow over a flat plate with a fitted to it triangular vibrator which effects harmonic oscillations is studied. The plate and vibrator are assumed heat-insulated, and the vibrator dimensions and oscillation frequency is such that the flow can be defined by equations of the boundary layer with self-induced pressure. The oscillation amplitude is assumed small, making it possible to linearize these equations. The solution is obtained by double application of the Fourier transform with respect to time and longitudinal coordinate. Inverse transformation is achieved by numerical methods. Analysis is carried out for the vibrator frequency ω lower than the critical ω_* predicted by the classical theory of stability. It is shown that vibrator-induced perturbations become rapidly damped upstream. Damping downstream is rapid for ω considerably lower than ω_* and slows down as ω approaches ω_*.

Consider the flow over a heat-insulated plate whose front part is at rest, followed by an oscillating section, the vibrator, and the end part is a stationary flat plate. Let the front part length be L^* and the rear part $O(L^*)$ (the asterisk denotes dimensional quantities). Let the unperturbed oncoming stream be subsonic with the Mach number M_∞ less than one by a finite quantity and velocity U_∞^* directed along the stationary parts of the body. The subscripts ∞ and w denote gas parameters in the unperturbed steady stream and at the wall, respectively. We use the Cartesian system of coordinates x, y with origin at the point of junction of the front stationary part with the vibrating part, and the following notation: t^* for time, v_x^*, v_y^* for the velocity vector components, ρ^* for density, p^* for pressure, T^* for temperature, and $\varkappa$ for the ratio of specific heats. For simplicity the dependence of the first viscosity coefficient on temperature is assumed to be locally linear (for $T^* \sim T_w^*$); $\lambda_1^*/\lambda_{1\infty}^* = CT'$, where $T' = T^*/T_\infty^*$, and the Prandtl number to be unity. Instead of the inverse value of the Reynolds number we use the small parameter $\varepsilon = \mathrm{Re}_1^{-1/8}$ $(\mathrm{Re}_1 = \rho_\infty^* U_\infty^* L^*/\lambda_{1\infty}^*)$.

We select the vibrator longitudinal dimension to be $O(L^*\varepsilon^3)$, the oscillation amplitude $O(L^*\varepsilon^5)$, and the oscillation frequency $O(U_\infty^*/L^*\varepsilon^2)$. For defining the motion generated by such vibrator it is convenient to separate three characteristic regions /1,2/, viz. the upper or external region of the inviscid subsonic flow $(y_1^* = O(L^*\varepsilon^3))$, the intermediate of the conventional boundary layer $(y_2^* = O(L^*\varepsilon^4))$, and the lower region of the boundary layer with self-induced pressure $(y_3^* = O(L^*\varepsilon^5))$. The basic difficulties in such scheme relate to the derivation of solution for the lower part. That solution enables us to obtain in explicit form the parameters of flow in the intermediate and upper regions /1—5/. Below we deal only with the lower region using the following dependent and independent variables /4,5/:

$$\begin{aligned}
t^* &= L^* U_\infty^{*-1} \varepsilon^2 C^{1/4} \lambda^{-3/2} (1 - M_\infty^2)^{-1/4} T_w' t \qquad (1)\\
x^* &= L^* \varepsilon^3 C^{3/8} \lambda^{-5/4} (1 - M_\infty^2)^{-3/8} T_w'^{3/2} x\\
y^* &= L^* \varepsilon^5 C^{5/8} \lambda^{-3/4} (1 - M_\infty^2)^{-1/8} T_w'^{3/2} y\\
v_x^* &= U_\infty^* \varepsilon C^{1/8} \lambda^{1/4} (1 - M_\infty^2)^{-1/8} T_w'^{1/2} u\\
v_y^* &= U_\infty^* \varepsilon^3 C^{3/8} \lambda^{3/4} (1 - M^2)^{1/8} T_w'^{1/2} v\\
p^* &= p_\infty^* + \rho_\infty^* U_\infty^{*2} \varepsilon^2 C^{1/2} \lambda^{1/2} (1 - M_\infty^2)^{-1/4} p\\
\rho^* &= \rho_\infty^* T_w' \rho.
\end{aligned}$$

where the constant $\lambda = 0.3321$ is determined by the equality $L^* \mathrm{Re}_1^{-1/2} \partial (v_x^*(L^*, 0)/U_\infty^*)/\partial y^* = \lambda C^{-1/2} T_w'$ from the Blasius solution for the unperturbed boundary layer. Substituting expressions (1) into the Navier—Stokes system of equations, retaining the principal terms in ε and

*Prikl.Matem.Mekhan.,45,No.6,1049-1055,1981 stipulating the fulfillment of conditions of merging as $x \to -\infty$ and $y \to \infty$, for the unsteady subsonic boundary layer with self-induced pressure we obtain the following system of equations and boundary conditions /4,5/:

$$\rho = 1, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$x \to -\infty, \quad u \to y, \quad p \to 0, \quad y \to \infty, \quad u \to y + \frac{1}{\pi}\int_{-\infty}^{x}\int_{-\infty}^{\infty} \frac{p}{x_1 - x_0}\, dx_1\, dx_0$$

In the last condition with $y \to \infty$ the inner integral is to be understood in the Cauchy sense of the principal value. We shall also seek a solution approaching the solution for the unperturbed boundary layer as $x \to \infty$, which, as shown below, is linked with the constraints imposed on the vibrator oscillation frequency.

We specify the condition of adhesion on the body as

$$u(t, x, y_w(t, x)) = u_w, \quad v(t, x, y_w(t, x)) = v_w \tag{3}$$

Let the oscillating part of the body be defined, as in the problem of the vibrator in a supersonic boundary layer /6/, by the equation

$$y_w = \sigma f(t, x) = \sigma f_1(x) \cos \omega t, \quad \sigma \ll 1, \quad \omega > 0 \tag{4}$$

where ω is the dimensionless frequency and function $f_1(x)$ defines the (vibrator) triangular form with parameters a and b ($f_1(x) = 0$ when $x \leqslant 0$, $2x$ when $0 \leqslant x \leqslant b$, $2b(a-x)/(a-b)$ when $b \leqslant x \leqslant a$, and 0 when $x \geqslant a$). The smallness of parameter σ enables us to linearize problem (2), (3) by expanding the solution in series in powers of σ

$$u = y + \sigma u_1 + \ldots, \quad v = \sigma v_1 + \ldots, \quad p = \sigma p_1 + \ldots$$

Equations and limit conditions determined by (2) and the condition as $x \to \infty$ are of the form

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \quad \frac{\partial p_1}{\partial y} = 0, \quad \frac{\partial u_1}{\partial t} + y\frac{\partial u_1}{\partial x} + v_1 = -\frac{\partial p_1}{\partial x} + \frac{\partial^2 u_1}{\partial y_2} \tag{5}$$

$$x_1 \to -\infty, \ u_1 \to 0, \ p_1 \to 0; \ x \to \infty, \ u_1 \to 0, \ p_1 \to 0, \quad y \to \infty, \quad u_1 \to \frac{1}{\pi}\int_{-\infty}^{x}\int_{-\infty}^{\infty} \frac{p_1(t, x_1)}{x_1 - x_0}\, dx_1\, dx_0$$

The adhesion condition (3) at wall (4) implies that

$$u_1(t, x, 0) = -f_1(x)\cos \omega t, \quad v_1(t, x, 0) = -f_1(x)\, \omega \sin \omega t \tag{6}$$

where only the principal terms are retained.

For solving problem (5), (6) we use the Fourier transform

$$\bar{u}_1(\omega_1, \omega_2, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\omega_1 t - i\omega_2 x}\, u_1(t, x, y)\, dt\, dx$$

Eliminating from system (5) v_1 and p_1 and passing from u_1 to $\bar{u}_1$, we obtain

$$\frac{\partial^3 \bar{u}_1}{\partial y^3} = (i\omega_2 y + i\omega_1)\frac{\partial \bar{u}_1}{\partial y}$$

The solution of this equation which satisfies the condition of boundedness of $\bar{u}_1$ as $y \to \infty$ is of the form

$$\frac{\partial \bar{u}_1}{\partial y} = B(\omega_1, \omega_2)\, \mathrm{Ai}\,[(i\omega_2)^{1/3} y + i^{1/3}\omega_1\omega_2^{-2/3}]$$

where Ai is the Airy function /7/, $i = \exp(i\pi/2)$, $B(\omega_1, \omega_2)$ is an arbitrary function of its arguments. The limit and boundary conditions (5) and (6) enable us to express $B(\omega_1, \omega_2)$ in terms of $\bar{f}(\omega_1, \omega_2)$ and obtain $\bar{p}_1(\omega_1, \omega_2)$, where $\bar{f}$ and $\bar{p}$ are the Fourier transforms $f(t, x)$ and $p_1(t, x)$, respectively. We have

$$\bar{p}_1 = |\omega_2|\, \mathrm{Ai}'(\Omega)\, \bar{f}(\omega_1, \omega_2)/Q(\Omega, \omega_2), \quad \Omega = \frac{i^{1/3}\omega_1}{\omega_2^{2/3}}, \quad I_0 = \int_0^{\infty} \mathrm{Ai}(x)\, dx = \frac{1}{3}, \tag{7}$$

$$I_1(\Omega) = \int_0^{\Omega} \mathrm{Ai}(z)\, dz, \quad Q(\Omega, \omega_2) = -\mathrm{Ai}'(\Omega) + i^{1/3}\omega_2^{1/3}\, |\omega_2|\, [I_0 - I_1(\Omega)]$$

where the prime denotes the derivative of Airy's function.

Let us calculate the pressure. The expression for p_1 is obtained using the inverse Fourier transform. Calculations similar to those in /6/ yield

$$p_1 = \frac{1}{\pi}\cos\omega t \int_{-\infty}^{\infty} \operatorname{Re}\Phi\, d\omega_2 - \frac{1}{\pi}\sin\omega t \int_{-\infty}^{\infty} \operatorname{Im}\Phi\, d\omega_2 \qquad (8)$$

$$\Omega_1 = \frac{i^{1/3}\omega}{\omega_2^{2/3}}, \quad \Phi = -\frac{e^{i\omega_2 x}}{|\omega_2|}\left(1 - \frac{a}{a-b}e^{-i\omega_2 b} + \frac{b}{a-b}e^{-i\omega_2 a}\right)\mathrm{Ai}'(\Omega_1)/Q(\Omega_1, \omega_2)$$

To select the single-valued branch of Φ in the complex plane ω_2 we make a slit from point O along the imaginary axis (Fig.1), i.e. $\pi/2 > \arg\omega_2 > -3\pi/2$. If in formulas (8) we make ω approach zero and assume parameters a and b to be considerably greater than unity, then for $|x| = O(1)$ the pressure will be equal to that near a corner at rest /8/.

As shown in /9,10/, the linear theory of boundary layer stability with respect to long-wave perturbations and the analysis of small perturbations in the theory of the boundary layer with self-induced pressure with subsonic external flow yield the same results. Thus, using the notation introduced above, when frequency $\omega < \omega_* \approx 2.298$, the perturbations are damped as $x \to \infty$, not damped when $\omega = \omega_*$, and when $\omega > \omega_*$ they increase with increasing x. Since in the limit condition (5) the unknown functions are required to approach zero as $x \to \infty$, which is necessary for the application of the Fourier transform, we confine the analysis to the range

$$0 < \omega < \omega_* \approx 2.298 \qquad (9)$$

Although the derived solution (8) formally exists also for $\omega > \omega_*$, we shall not consider such ω, since the question of existence of a steady (with respect to time) mode at such ω requires a separate investigation.

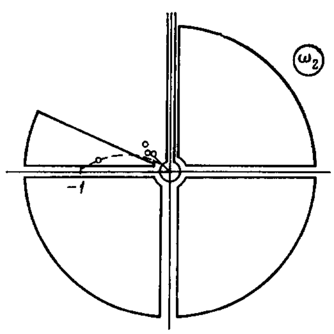

Fig.1

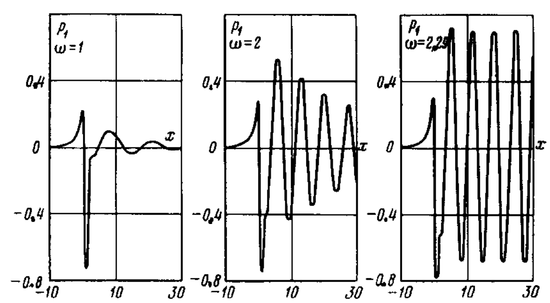

Fig.2

An integral of Φ obviously exists for any x, since with $|\omega_2| \to \infty$ the integrand is proportional to $|\omega_2|^{-7/3}$ when $|\omega_2| \to 0$ and $\arg\omega_2 = 0$; the π integrand is bounded, and the inequality (9) ensures the absence of real roots in the denominator (9,10/. The basic complexities of the calculation of pressure p_1 is related to the nonanalyticity of function Φ. Unlike in the supersonic case /11/, it is not possible here to represent the result of calculations in the form of everywhere converging series in powers of $x^{1/3}$. It can be shown that for a subsonic flow even in the steady case $(\omega = 0)$ logarithmic terms appear in the series. Calculation of pressure p_1 defined in (8) was carried out in two stages. First, the integral of the derivative $d\Phi/dx$ was determined. For this the derivative was divided in three terms proportional to $\exp(i\omega_2 x_1)$ (where $x_1 = x, x-b, x-a$), calculation of the integral of each term was effected over the most convenient integration path: for $x_1 \leqslant 0$ over the imaginary negative semiaxis; for $x_1 > 0$ over the imaginary positive semiaxis, and for $x_1 > 0$ it was also necessary to summate the series formed by residues of integrands. In the second stage integration was carried out with respect to x, and pressure p_1 was determined. Curves of the dependence of pressure p_1 on x in the case of a triangle with parameters $b = 1, a = 2$ oscillating at frequencies $\omega = 1; 2; 2.290$ at the instant of time $t = 0$ appear in Fig.2.

Let us consider the asymptotic behavior of pressure at large x, beginning with the case of $x \to -\infty$. We represent the integral of Φ in the form

$$\int_{-\infty}^{\infty} \Phi\, d\omega_2 = I_2 + I_3, \quad I_2 = \int_{-\infty}^{0} \Phi\, d\omega_2, \quad I_3 = \int_{0}^{\infty} \Phi\, d\omega_2 \tag{10}$$

then in expressions for I_2 and I_3 under the integral sign we have analytic functions. To calculate I_2 we use the contour lying in the third quadrant and consisting of segments $[-r, 0]$ and $[0, -ir]$, and the arc of circle connecting points $-ir$ and $-r$ (Fig.1). Inside the part of the complex plane enveloped by the above contour the denominator of the integrand $Q(\Omega_1, \omega_2)$ for any $r > 0$ and ω that satisfy the inequality (9) has no roots; the integral along the arc of circle approaches zero $(x < 0)$ as $r \to \infty$, hence integration along the real axis in I_2 (10) can, in conformity with the Cauchy theorem, be replaced by integration along the imaginary axis from $\infty \exp(-i\pi/2)$ to zero. For the similar transformation of I_3 we use the contour that is symmetric to that for I_2 about the imaginary axis. Then, carrying out transformations associated with the substitution $\omega_2 = \omega_3 \exp(-i\pi/2)$, as $x \to -\infty$ we have

$$\int_{-\infty}^{\infty} \Phi\, d\omega_2 \sim ab \Big\{ x^{-2} + \frac{2}{3}(a+b)x^{-3} + \tag{11}$$
$$\frac{1}{2}(a^2 + ab + b^2)x^{-4} + \frac{2}{5}(a+b)(a^2+b^2)x^{-5} +$$
$$\Big[\frac{1}{3}(a^4 + a^3b + a^2b^2 + ab^3 + b^4) + 120\omega^{-2}\Big]x^{-6} + \ldots \Big\}$$

Formula (11) shows that the asymptotics is not uniform with respect to ω: as $\omega \to 0$ the coefficient in the fifth expansion term approaches infinity. Analysis of asymptotics in the steady flow $(\omega = 0)$ shows that as $x \to -\infty$ the first three terms are the same as in (11), while the fourth is proportional to $(-x)^{-14/3}$.

Consider the asymptotics of pressure as $x \to \infty$. To evaluate integral I_2 (10) we shall analyze the integrand in the second quadrant, where the denominator of expression $Q(\Omega_1, \omega_2)$ under the sign of the integral has an infinite denumerable set of roots. Although the position of roots depends on ω, all of them, except the first lie near the half-line $\omega_2 = |\omega_2| \exp(-5i\pi/4)$ and have $\omega_2 = 0$ as the concentration point (Fig.1). The first root ω_{21} as ω increases from 0 to ω_* moves away from the half-line $\omega_2 = |\omega_2| \exp(-5i\pi/4)$ and reaches the real axis $\omega_{21}(\omega_*) = -1.0005$. Its trajectory is shown in Fig.1 by the dash line. The analysis of roots shows that for ω from the range (9) it is possible to draw from the coordinate origin a half-line at such angle $\alpha(\omega)$ $(-\pi > \alpha > -5\pi/4)$ that the first root lies on one side of the half-line and all others on the other. Let us take the contour lying in the second quadrant and consisting of segments $[-r, 0]$, $[0, r\exp(i\alpha)]$ and the arc of circle connecting points $-r$ and $r\exp(i\alpha)$ (Fig.1). Inside that contour lies a first order pole of the integrand of the expression of I_2 and the integral over the arc of circle approaches zero as $r \to \infty$ and $x > a$. Applying the Cauchy theorem on residues and introduce in the integral over the half-line the substitution of variables $\omega_2 = \omega_3 \exp(i\alpha)$ and also $\alpha_1 = -\pi - \alpha$ $(0 < \alpha_1 < \pi/4)$, we obtain

$$I_2 = 2\pi i \operatorname{res} \Phi(\omega, \omega_{21}(\omega), x, b, a) + \tag{12}$$
$$\int_0^\infty \frac{e^{\varphi_1(x)}}{\omega_3}\Big(1 - \frac{a}{a-b}e^{-\varphi_1(b)} + \frac{b}{a-b}e^{-\varphi_1(a)}\Big)\mathrm{Ai}'(\Omega_3) \times$$
$$[\mathrm{Ai}'(\Omega_3) + (-7\exp i\pi/6 - 4i\alpha_1/3)\,\omega_3^{4/3}(I_0 - I_1(\Omega_3))]^{-1}\, d\omega_3$$
$$2\pi i \operatorname{res} \Phi(\omega, \omega_{21}(\omega), x, b, a) = B_1(\omega, \omega_{21}(\omega), b, a) \times \exp(i\omega_{21}(\omega)x)$$
$$B_1 = -3\pi e^{i\pi/3}\omega_{21}^{-4/3}\Big(1 - \frac{a}{a-b}e^{-i\omega_{21}b} + \frac{b}{a-b}e^{-i\omega_{21}a}\Big)\mathrm{Ai}'(\Omega_{11}) \times$$
$$[2(I_0 - I_1(\Omega_{11})) + \Omega_{11}(1 - \omega/\omega_{21}^2)\,\mathrm{Ai}(\Omega_{11})]^{-1},$$
$$\Omega_{11} = i^{1/3}\omega\omega_{21}^{-2/3}$$
$$\varphi_1(x) = -\omega_3 x \sin\alpha_1 - i\omega_3 x \cos\alpha_1, \quad \Omega_3 = \omega\omega_3^{-2/3}\exp(5i\pi/6 + 2i\alpha_1/3)$$

To evaluate integral I_3 we consider the integrand in the first quadrant, where the denominator of integrand $Q(\Omega_1, \omega_2)$ has no roots. We select the contour consisting of segments $[r\exp(i\pi/2), 0]$, $[0, r]$ and the arc of circle connecting points r and $r\exp(i\pi/2)$. Since the integral over the arc of a circle approaches zero as $r \to \infty$, the upper limit of integration in I_3 can be changed to $\infty \exp(i\pi/2)$, and the imaginary axis taken as the integration path. Introducing in such integral the substitution of variables $\omega_2 = \omega_3 \exp(i\pi/2)$, we obtain

$$I_3 = \int_0^\infty \frac{1}{\omega_3} e^{-\omega_3 x} \left(1 - \frac{a}{a-b} e^{\omega_3 b} + \frac{b}{a-b} e^{\omega_3 a}\right) \mathrm{Ai}'(\Omega_4) \times [\mathrm{Ai}'(\Omega_4) - i^{5/3} \omega_3^{4/3} (I_0 - I_1(\Omega_4))]^{-1} d\omega_3, \quad \Omega_4 = i^{-1/3} \omega \omega_3^{-2/3} \tag{13}$$

The integral I_2 in (12) as well as integral I_3 in (13) have been reduced to the form suitable for the application of Laplace's lemma /12/. Retaining in the expression for the asymptotics as $x \to \infty$ also the term generated by the pole of Φ, we obtain

$$\int_{-\infty}^{\infty} \Phi \, d\omega_2 \sim ab \Big\{x^{-2} + \frac{2}{3}(a+b)x^{-3} + \frac{1}{2}(a^2+ab+b^2)x^{-4} + \frac{2}{5}(a+b)(a^2+b^2)x^{-5} + \tag{14}$$

$$\left[\frac{1}{3}(a^4 + a^3b + a^2b^2 + ab^3 + b^4) + 120\omega^{-2}\right] x^{-6}\Big\} + B_1 e^{i\omega_{21}(\omega)x}$$

Asymptotics (14) coincide within the exponential terms with asymptotics (11) and is non-uniform with respect to ω: as $\omega \to 0$ the coefficient at the fifth expansion term approaches infinity. Analysis of the asymptotics of the steady flow $(\omega = 0)$ shows that, as $x \to \infty$, only the first two terms are the same as in (14), while the third term is proportional to $x^{-10/3}$. At large x and $\omega \to \omega_*$ $(\omega < \omega_*)$ the determining term in (14) is the exponential one, since then $\mathrm{Im}\, \omega_{21} \to 0$ (note the absence of a similar term in expansion (11)).

The derived solution defines the perturbations induced in the boundary layer by the oscillating vibrator. As the oscillation frequency ω approaches the critical value ω_* predicted by the classical theory of stability; the closer is ω to ω_* the slower is the damping of oscillations downstream /of the oscillator/, (Fig.2) and the law of damping $(\exp(-x \,\mathrm{Im}\, \omega_{21}(\omega)))$ depends only on frequency ω. The initial oscillation amplitude is defined by the constant B_1 dependent on the specific form and dimensions a and b of the oscillator. Upstream $(x < 0)$ the closeness of ω to ω_* does not manifest itself in any way (Fig.2, $\omega = 2.29$).

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